

**AD-A256 526**



CMS Technical Summary Report #93-4

**MULTIRESOLUTION ANALYSIS BY  
INFINITELY DIFFERENTIABLE  
COMPACTLY SUPPORTED FUNCTIONS**

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September 1992

(Received September 25, 1992)

**92-27853**



12p8

**Approved for public release  
Distribution unlimited**

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

National Science Foundation  
1800 G. Street  
Washington, DC 20550

# Multiresolution analysis by infinitely differentiable compactly supported functions

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## ABSTRACT

The paper is concerned with the introduction and study of multiresolution analysis based on the  $\text{up}$  function, which is an infinitely differentiable function supported on  $[0, 2]$ . Such analysis is, necessarily, nonstationary. It is shown that the approximation orders associated with the corresponding spaces are spectral, thus making the spaces attractive for the approximation of very smooth functions.

AMS (MOS) Subject Classifications: 41A25 41A30 46C99 39B99

Key Words:  $\text{up}$  function, shift-invariant spaces, wavelets, multiresolution analysis, spectral approximation orders, stability.

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Supported in part by the Israel-U.S. Binational Science Foundation (grant 9000220), by the U.S. Army (Contract DAAL03-G-90-0090), and by the National Science Foundation (grants DMS-9000053, DMS-9102857).

# Multiresolution analysis by infinitely differentiable compactly supported functions

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## 1. Introduction

Multiresolution analysis based on a compactly supported refinable function is limited to generators with a finite degree of smoothness. In this paper we discuss multiresolution analysis with  $C^\infty$ -compactly supported generators. This is possible if the generator of each space is related to the generator of the next finer space by a mask whose support grows linearly with the resolution of the space. We consider here a particular instance of such analysis based on the up function of Rvachev [Rv], defined as follows.

Let

$$\chi_k(x) := \begin{cases} 2^k, & 0 \leq x \leq 2^{-k}, \\ 0, & \text{otherwise,} \end{cases}$$

and let  $\sigma$  denote the infinite convolution product

$$(1.1) \quad \sigma := \chi_1 * \chi_2 * \dots$$

Then the up function, denoted here by  $\phi_0$ , is given by

$$\phi_0 = \chi_0 * \sigma.$$

It follows from its definition that the up function is supported on the interval  $[0, 2]$  and is infinitely smooth. Approximation properties of this function can be found in [Rv]. In [DDL] it is shown that the up function can be obtained as a limit of a non-stationary subdivision scheme, which employs at a level  $k$  the mask of the stationary scheme that corresponds to the B-spline of degree  $k$ . As observed in [DDL], this is equivalent to the existence of a family of functions  $(\phi_k)$  satisfying an infinite system of functional equations relating each  $\phi_k$  to (appropriately scaled) shifts of  $\phi_{k+1}$  in terms of the above mentioned mask.

In Section 2 we introduce the relevant functions  $(\phi_k)_{k=0}^\infty$  (the first of which is the above mentioned up function), and use them to define the ladder of spaces  $S_0 \subset S_1 \subset S_2 \subset \dots$  with each  $S_k$  being the "span" of the  $2^{-k}\mathbb{Z}$ -shifts of the corresponding  $\phi_k$ . The resolution obtained in this way is **nonstationary** in the sense that  $\phi_k$  is not a dilate of its predecessor. We provide the wavelet decomposition, based on the general theory of nonstationary multiresolution analysis of [BDR2]. We discuss the stability issue and show that the generator we choose for each wavelet space is stable (and even linearly independent), but that the  $L_2$ -stability bounds blow up with  $k$  at the rate (no faster than)  $(\pi/2)^{3k}$ . On the other hand we show in Section 3, using the general theory in [BDR1], that the least square approximation from  $(S_k)$  is spectral, namely that, for any  $r \geq 0$ , the  $L_2(\mathbb{R})$ -error of best approximation to  $f \in W_2^r$  from  $S_k$  is  $o(2^{-rk})$ . This means that high resolution of a very smooth  $f$  can be achieved for a relatively small  $k$ , and in such a case the difficulty of the growth of the stability constants may be less of a problem. It is thus possible that the multiresolution analysis based on the up function that we present here might be practically suitable for the decomposition of very smooth functions.

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In the paper, we use, for a compactly supported function  $f : \mathbb{R} \rightarrow \mathbb{C}$ , and a function  $g$  defined (at least) on  $2^{-k}\mathbb{Z}$ , the notation

$$(1.2) \quad f *_k' g := \sum_{j \in 2^{-k}\mathbb{Z}} f(\cdot - j)g(j).$$

## 2. Wavelet decompositions

The up function provides an interesting example of wavelet decompositions via multiresolution. A general discussion of these topics can be found in [BDR2], and is certainly beyond the scope of this paper.

A **multiresolution** begins with a sequence  $(\phi_k)_{k \in \mathbb{Z}_+} \subset L_2(\mathbb{R})$ . For each  $k$ , one denotes by  $S_k$  the smallest closed  $L_2$ -subspace that contains all the functions

$$(2.1) \quad \phi_k(\cdot - j), \quad j \in 2^{-k}\mathbb{Z}.$$

The nestedness assumption

$$(2.2) \quad S_k \subset S_{k+1}, \quad k \in \mathbb{Z}_+$$

is in the essence of the process. With (2.2) in hand, one defines the wavelet space  $W_k$  to be the orthogonal complement of  $S_k$  in  $S_{k+1}$ ,

$$(2.3) \quad W_k := S_{k+1} \ominus S_k.$$

Under mild conditions on the sequence  $(\phi_k)$  (cf. section 4 of [BDR2]; these conditions are always satisfied for compactly supported  $(\phi_k)$ , which will be the case here) the wavelet spaces provide an orthogonal decomposition of  $L_2(\mathbb{R}) \ominus S_0$ , and the subsequent task is then to find efficient (more precisely stable) methods for computing the orthogonal projection  $P_k f$  of a given  $f \in L_2(\mathbb{R})$  on each of the wavelet spaces. The attraction in these decompositions is that (in many examples) the information on  $f$  recorded by  $P_k f$  is considered to be “finer” as  $k$  increases.

In the original formulation of multiresolution, [Ma], [Me], it was assumed that the ladder  $(S_k)$  is **stationary**, namely, that each  $S_{k+1}$  is the 2-dilate of  $S_k$ . Analysis of nonstationary decompositions can be found in [BDR2], with the guiding example there being exponential B-splines. The wavelet decompositions that correspond to the up function are nonstationary as well, still they form a different variant in this class. We will elaborate on that point in the sequel.

Our sequence  $(\phi_k)_k$  can be defined as follows. First, we recall the definition of  $\sigma$  given in (1.1). With  $B_k$  the cardinal B-spline of degree  $k$  (i.e., with integer breakpoints and with support  $[0, k+1]$ ), we choose

$$(2.4) \quad \phi_k := (B_k * \sigma)(2^k \cdot).$$

Note that  $\text{supp } \phi_k = [0, (k+2)/2^k]$ . The spaces  $(S_k)$  are defined as in the beginning of this section, with respect to the present choice of  $(\phi_k)$ . Our discussion here is developed in two steps. First, we will observe below that the spaces  $(S_k)$  satisfy the nestedness assumption (2.2). Knowing therefore that the corresponding wavelets spaces  $(W_k)_{k \in \mathbb{Z}}$  are well-defined, we will then consider the problem of finding stable generators for the associated wavelet spaces.

Since

$$(\chi_k * f)(2 \cdot) = \chi_{k+1} * (f(2 \cdot)),$$

and

$$B_k = \underbrace{\chi_0 * \dots * \chi_0}_{(k+1)\text{-times}},$$

we find that

$$\phi_k := \underbrace{\chi_k * \dots * \chi_k}_{(k+1)\text{-times}} * \chi_{k+1} * \dots$$

Substituting  $k-1$  for  $k$  we get

$$\phi_{k-1} = \underbrace{\chi_{k-1} * \dots * \chi_{k-1}}_{k\text{-times}} * \chi_k * \chi_{k+1} \dots$$

Therefore, a **refinement equation** that expresses  $\phi_{k-1}$  as a linear combination of the translates of  $\phi_k$  will be the same as the one that connects the splines

$$\underbrace{\chi_{k-1} * \dots * \chi_{k-1}}_{k\text{-times}}, \text{ and } \underbrace{\chi_k * \dots * \chi_k}_{k\text{-times}}.$$

These splines are (up to the factors  $2^{k-1}$  and  $2^k$  respectively) scales of the B-spline  $B_{k-1}$ , and the refinement equation becomes identical to the well-known one for B-splines. Indeed, the solution  $A_k$  for the convolution equation

$$\phi_{k-1} = \phi_k *'_k A_k$$

is the  $k$ -fold convolution product of the sequence

$$A_k^0(j) := \begin{cases} 1/2, & j = 0, 2^{-k}, \\ 0, & j \in 2^{-k}(\mathbb{Z} \setminus \{0, 1\}), \end{cases}$$

that solves the equation

$$\chi_{k-1} = \chi_k *'_k A_k^0.$$

Its Fourier transform has the form

$$\hat{A}_k(w) = \left( \frac{1 + e^{-iw/2^k}}{2} \right)^k.$$

To conclude, in terms of Fourier transforms we obtained the following refinement equation:

**Corollary 2.5.**

$$\widehat{\phi}_{k-1} = \widehat{A}_k \widehat{\phi}_k.$$

Since  $A_k$  is finitely supported, the corollary shows that  $\phi_{k-1}$  can be expressed as a finite linear combination of  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_k$ , thereby proving the required nestedness property. Consequently, the corresponding wavelets spaces are well-defined. Note that, while  $(\phi_k)_k$  satisfies the same refinement equations that are being satisfied by B-splines, the degree of the associated B-spline changes with  $k$ . In particular, the size of the support of the mask sequence  $A_k$  grows linearly with  $k$ . In contrast, the nonstationary decompositions associated with exponential B-splines (cf. [BDR2: §6], [DL]) employ masks with uniformly bounded support.

Corollary 2.5 allows us to apply standard wavelet techniques (cf. [CW], [JM], [BDR2]). In particular, it is known [BDR2] that the function  $\psi_k$  defined by

$$(2.6) \quad \widehat{\psi}_{k-1}(w) := e^{-iw/2^k} \overline{\widehat{A}_k(w + 2^k\pi)} \tau_k(w) \widehat{\phi}_k(w),$$

with

$$(2.7) \quad \tau_k(w) := \sum_{j \in 2\pi 2^k \mathbb{Z}} |\widehat{\phi}_k(w + j)|^2$$

generates  $W_k$  in the sense that the  $2^{-k}\mathbb{Z}$ -shifts of  $\psi_k$  are fundamental in  $W_k$ . A standard application of Poisson's summation formula yields that  $\tau_k$  is a trigonometric polynomial with frequencies in  $(2^{-k}\mathbb{Z}) \cap [-(k+1)/2^k, (k+1)/2^k]$ . Some straightforward computation then implies that  $\psi_{k-1}$  is supported in an interval of length  $(k+1)/2^{k-2}$  which is exactly twice the size of the support of  $\phi_{k-1}$ .

Now, we turn our attention to the *stability* question. The generator  $\psi_k$  is called **stable** if the restriction  $R_k$  of  $\psi_k \star'_k$  to  $\ell_2(\mathbb{Z})$  is well-defined, bounded and boundedly invertible. Since the decomposition here is nonstationary, it is also important to make sure that the norms  $\|R_k\|$  and  $\|R_k^{-1}\|$  are bounded independently of  $k$  (cf. [JM] and [BDR3] for detailed discussion of the stability problem).

It is known (cf. section 5 of [BDR2] and especially Remark 5.8 there) that  $\psi_k$  is a stable generator of  $W_k$  if each  $\phi_{k'}$  is a stable generator of  $S_{k'}$ ,  $k' \in \mathbb{Z}_+$ , and further,  $\|R_k\|$  and  $\|R_k^{-1}\|$  are bounded by rational expressions in  $\|T_{k'}\|$ ,  $\|T_{k'}^{-1}\|$ ,  $k' = k, k+1$ , with  $T_k$  being the restriction to  $\ell_2(\mathbb{Z})$  of  $\phi_k \star'_k$ , and with the rational expressions being independent of  $k$ :

**Proposition 2.8.**  $\psi_k$  in (2.6) is a stable generator of  $W_k$  if  $\phi_{k'}$ ,  $k' = k, k+1$ , is a stable generator of  $S_{k'}$ . Further, the stability constants associated with  $(\psi_k)_k$  are uniformly bounded if the same holds for the stability constants of  $(\phi_k)_k$ , since

$$\|R_k\| \|R_k^{-1}\| \leq \text{const} \|T_k\| \|T_k^{-1}\| (\|T_{k+1}\| \|T_{k+1}^{-1}\|)^2.$$

We prove that each  $\phi_k$  is a stable generator of  $S_k$  with the aid of the following well-known result (cf. [SF], [DM], [JM] and [BDR3]).

**Result 2.9.** Let  $\phi$  be a compactly supported  $L_2(\mathbb{R})$ -function. Then the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi$  are  $L_2$ -stable if and only if for every  $\theta \in \mathbb{R}$ , there exists  $j \in 2^{k+1}\pi\mathbb{Z}$  such that  $\hat{\phi}(\theta + j) \neq 0$ . In other words, the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi$  are stable if and only if  $\hat{\phi}$  does not have a  $2^{k+1}\pi\mathbb{Z}$ -periodic zero.

We show below that the entire function  $\hat{\phi}_k$  has no  $2^{k+1}\pi\mathbb{Z}$ -periodic zero in the complex domain  $\mathbb{C}$ . This property is known to be equivalent to  $\phi_k \star'_k$  being *injective*, a property which is usually referred to as the **linear independence** of the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_k$  (cf. [Ro] for details). In view of Result 2.9, this will certainly imply that  $\phi_k$  is a stable generator of  $S_k$ .

**Corollary 2.10.** The  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_k$  are linearly independent, hence form a stable basis for the space  $S_k$  that they generate. However, the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_{k'}$  are not  $L_2$ -stable, whenever  $k' < k$ .

**Proof.** By Corollary 2.5, we have

$$\hat{\phi}_{k'} = \hat{A}_{k'+1} \hat{\phi}_{k'+1}.$$

Since  $\hat{A}_{k'+1}$  is  $2^{k'+2}\pi$ -periodic and  $k' < k$ , it is also  $2^{k+1}\pi$ -periodic, hence, in view of Result 2.9, the stability of the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_{k'}$ ,  $k' < k$ , forces  $A_{k'+1}$  to have no zeros (on  $\mathbb{R}$ ). However,  $\hat{A}_{k'+1}$  vanishes at  $2^{k'+1}\pi$ , and this proves the second statement of the corollary.

For the linear independence claim, we first remark that basic convergence criteria (cf. e.g. Theorem 15.4 of [Ru]) show that  $\hat{\phi}_k$  vanishes at a point (if and) only if one of its factors  $\hat{\chi}_j$  vanishes there. Secondly, we observe that since  $\hat{\chi}_j(w) = 2^j \int_0^{2^{-j}} e^{-i\omega t} dt$ ,

$$\hat{\chi}_j(w) = 0 \iff w \in 2^{j+1}\pi\mathbb{Z} \setminus 0.$$

In particular, for  $j < j'$ ,  $\hat{\chi}_j(w) = 0$  if  $\hat{\chi}_{j'}(w) = 0$ . Since  $\hat{\phi}_k$  is the product of factors of the form  $\hat{\chi}_j$ , for  $j \geq k$ , we conclude that the (complex) zeros of  $\hat{\phi}_k$  are identical with these of  $\hat{\chi}_k$ . Since  $\text{supp } \chi_k = [0, 2^{-k}]$ , the  $2^{-k}\mathbb{Z}$ -shifts of this function are trivially linearly independent, hence  $\hat{\chi}_k$  cannot have a  $2^{k+1}\pi\mathbb{Z}$ -periodic zero. Therefore,  $\hat{\phi}_k$  does not have such a zero, and consequently its  $2^{-k}\mathbb{Z}$ -shifts are linearly independent as well.  $\spadesuit$

While the  $2^{-k}\mathbb{Z}$ -shifts of  $\phi_k$  are linearly independent hence stable, the stability constants are not uniformly bounded. This assertion is based on the next result, which also provides some estimate on the growth of these constants as  $k \rightarrow \infty$ .

**Proposition 2.11.** Let  $T_k$  denote the restriction of  $\phi_k \star'_k$  to  $\ell_2(\mathbb{Z})$ . Then

- (a)  $\|T_k\| \|T_k^{-1}\| \xrightarrow{k \rightarrow \infty} \infty$ .
- (b)  $\|T_k\| \|T_k^{-1}\| \leq \text{const}(\pi/2)^k$ .

**Proof.** We recall ([BDR2]) that

$$(2.12) \quad \|T_k\|^2 \|T_k^{-1}\|^2 = \frac{\sup_{w \in \mathbb{R}} \sum_{\alpha \in 2^{k+1}\pi\mathbb{Z}} |\hat{\phi}_k(w + \alpha)|^2}{\inf_{w \in \mathbb{R}} \sum_{\alpha \in 2^{k+1}\pi\mathbb{Z}} |\hat{\phi}_k(w + \alpha)|^2} = \frac{\sup_{w \in \mathbb{R}} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\hat{B}_k \hat{\sigma})(w + \alpha)|^2}{\inf_{w \in \mathbb{R}} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\hat{B}_k \hat{\sigma})(w + \alpha)|^2}.$$

Since, for each  $k$ ,  $\widehat{B}_k(0) = 1$ , and also  $\widehat{\sigma}(0) = 1$ , we conclude that the numerator of the most right expression in (2.12) is  $\geq 1$ . As for the denominator, we observe that

$$|\widehat{B}_k(w)| = \left( \frac{|e^{-iw} - 1|}{|w|} \right)^k,$$

and that

$$\frac{|e^{-iw} - 1|}{|w|} < 1 \quad \text{on } \mathbb{R} \setminus 0.$$

This means that, for any fixed  $0 \neq |w| \leq \pi$ , and any  $\alpha \in 2\pi\mathbb{Z}$ , the non-negative sequence  $(|(\widehat{B}_k\widehat{\sigma})(w + \alpha)|^2)_k$  converges to zero monotonely as  $k \rightarrow \infty$ . Therefore, for such  $w$ ,

$$(2.13) \quad \lim_{k \rightarrow \infty} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\widehat{B}_k\widehat{\sigma})(w + \alpha)|^2 = 0.$$

This proves (a).

Since we have just proved that  $\sum_{\alpha \in 2\pi\mathbb{Z}} |(\widehat{B}_k\widehat{\sigma})(w + \alpha)|^2$  is non-increasing with  $k$ , we can estimate

$$\sup_{w \in \mathbb{R}} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\widehat{B}_k\widehat{\sigma})(w + \alpha)|^2 \leq \sup_{w \in \mathbb{R}} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\widehat{B}_0\widehat{\sigma})(w + \alpha)|^2 = \text{const.}$$

In addition, for  $w \in [-\pi, \pi]$ , we estimate

$$\inf_{w \in \mathbb{R}} \sum_{\alpha \in 2\pi\mathbb{Z}} |(\widehat{B}_k\widehat{\sigma})(w + \alpha)|^2 \geq |\widehat{\sigma}(w)|^2 \left( \frac{|e^{-iw} - 1|}{|w|} \right)^{2k} \geq \text{const} \left( \frac{2}{\pi} \right)^{2k},$$

where, in the last inequality, we have used the facts that (i):  $\widehat{\sigma}$  vanishes nowhere on  $[-\pi, \pi]$ , (ii): the minimal value of  $\frac{|e^{-iw} - 1|}{|w|}$  (assumed at  $w = \pi$ ) is  $2/\pi$ .

Claim (b) is then obtained by combining the estimates above.  $\spadesuit$

From Proposition 2.8, we conclude that the stability constants associated with the wavelets  $(\psi_k)_k$  grow no faster than  $O((\pi/2)^{3k})$ .

### 3. Approximation Orders

As a natural continuation of the previous discussion, we consider  $L_2$ -approximation orders of the spaces  $(S_k)$  generated by  $(\phi_k)$  of (2.4). Given  $r > 0$ , let  $W_2^r$  be the usual potential space

$$(3.1) \quad W_2^r := \{f \in L_2(\mathbb{R}) : \|f\|_{W_2^r} := (2\pi)^{-1/2} \|(1 + |\cdot|)^r \widehat{f}\|_{L_2(\mathbb{R})} < \infty\}.$$

We say that  $(S_k)_k$  (or,  $(\phi_k)_k$ ) has approximation order  $r > 0$  (in the  $L_2$ -norm) if

$$(3.2) \quad E_k(f) := \text{dist}_{L_2(\mathbb{R})}(f, S_k)$$

satisfies

$$(3.3) \quad E_k(f) \leq \text{const}_r \|f\|_{W_2^r} 2^{-kr}, \quad \forall f \in W_2^r, \quad \forall \text{ sufficiently large } k.$$



here. "sufficiently large" may depend on  $r$ , but not on  $f$ . If, in addition to the above,

$$(3.4) \quad 2^{kr} E_k(f) \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad \forall f \in W_2^r,$$

we say that  $(S_k)$  has **density order**  $r$ . The notion of density orders extends to  $r = 0$ ; in such a case the definition is reduced to the requirement that

$$E_k(f) \rightarrow 0, \quad \forall f \in L_2(\mathbb{R}).$$

Although it is not obvious from the definition,  $(\phi_k)_k$  has all approximation orders  $\leq j$  whenever it has the approximation order  $j$ . We will show that the approximation by  $(S_k)_k$  is **spectral** which means, by definition, that  $(\phi_k)_k$  has all positive approximation orders.

Our precise result is as follows:

**Theorem 3.5.** *The sequence of spaces  $(S_k)$  has density order  $r$  for every  $r \geq 0$ .*

The fact that all *approximation* orders are obtained is plausible, since each  $\phi_k$  is obtained by smoothing  $B_k(2^k \cdot)$ , and for a fixed  $j$ , the functions  $(B_j(2^k \cdot))_k$  are well-known to have approximation order  $j + 1$ . However, the fact that we obtain even all *density* orders seems to be less expected. Furthermore, in what follows we show that for a very smooth function  $f$  (whose Fourier transform decays exponentially),  $E_k(f)$  decays exponentially, as well.

A general comprehensive discussion of  $L_2$ -approximation orders and density orders for shift invariant spaces is given in [BDR1]. Instead of deriving Theorem 3.5 from those results, we will apply the approach taken there to our special (and much simpler) case. This will result at tighter estimates for  $E_k(f)$ . Similar results in other  $p$ -norms are available as well. For example, spectral approximation in the uniform norm can be proved by employing the results of [BR].

Our analysis of the approximation orders goes as follows: let  $I = [-t, t] \subset [-\pi, \pi]$ . In order to estimate  $E_k(f)$ , we localize  $f$  on the Fourier domain. Precisely, we multiply  $\hat{f}$  by the characteristic function  $\eta_k$  of the interval  $2^k I$ , to obtain the function

$$(3.6) \quad g_k := (\eta_k \hat{f})^\vee,$$

and use the straightforward bound

$$(3.7) \quad E_k(f) \leq E_k(g_k) + \|((1 - \eta_k) \hat{f})^\vee\|.$$

We refer to the first term in the above sum as the **approximation error**, or the **projection error**, and to the second term as the **truncation error**.

The decay rate of the truncation error is clearly independent of  $(\phi_k)$ , and depends on the smoothness class of  $f$ . In particular, the following can be easily proved (cf. [BDR1]):

**Lemma 3.8.** *Let  $I := [-t, t]$  be some neighborhood of the origin, and let  $\eta_k$  be the characteristic function of  $2^k I$ . Let  $f \in W_2^r$ ,  $r \geq 0$ . Then*

$$\|((1 - \eta_k) \hat{f})^\vee\|_{L_2(\mathbb{R})} \leq t^{-r} 2^{-kr} \|f\|_{W_2^r} \varepsilon_k(f, t),$$

where  $0 \leq \varepsilon_k(f, t) \leq 1$  and converges to 0 as  $k$  tends to  $\infty$ .

In view of Lemma 3.8, a proof of Theorem 3.5 requires the study of the behaviour of the projection error. To estimate this, we employ the following result from [BDR1]. In what follows we use, for  $f \in L_2(\mathbb{R})$ , the notation  $S(f)$  to denote the smallest closed subspace of  $L_2(\mathbb{R})$  that contains all the shifts of  $f$ .

**Result 3.9.** Let  $\xi$  be a function in  $L_2$  and let  $g \in L_2$  be a function whose Fourier transform  $\widehat{g}$  is supported in  $I \subset [-\pi, \pi]$ . Let  $Pg$  be the orthogonal projection of  $g$  on  $S(\xi)$ . Then

$$\|g - Pg\|_{L_2(\mathbb{R})} = (2\pi)^{-1/2} \|\widehat{g} \Lambda_\xi\|_{L_2(I)},$$

where

$$\Lambda_\xi^2 := 1 - \frac{|\widehat{\xi}|^2}{\sum_{j \in 2\pi\mathbb{Z}} |\widehat{\xi}(\cdot + j)|^2}.$$

We intend to apply the last result to the function  $g := g_k(2^{-k}\cdot)$ , with  $g_k$  as in (3.6). We first note that, for any  $f \in L_2(\mathbb{R})$ , by dilating,

$$E_k(f) = 2^{-k/2} \text{dist}(f(2^{-k}\cdot), S(\phi_k(2^{-k}\cdot))).$$

Thus, we can use Result 3.9 with respect to  $\xi := \phi_k(2^{-k}\cdot) = B_k * \sigma$ . We estimate  $\Lambda_\xi$  as follows. We first denote

$$M_\xi^2 := \sum_{j \in 2\pi\mathbb{Z} \setminus 0} |\widehat{\xi}(\cdot + j)|^2.$$

Then

$$\Lambda_\xi^2 = \frac{M_\xi^2}{|\widehat{\xi}|^2 + M_\xi^2} \leq \frac{M_\xi^2}{|\widehat{\xi}|^2}.$$

Further,

$$|\widehat{\xi}(w)| = |\widehat{B}_k(w)| |\widehat{\sigma}(w)| = |\tau_k(w)| |w|^{-k} |\widehat{\sigma}(w)|,$$

with  $\tau_k$  a  $2\pi$ -periodic trigonometric polynomial. We conclude that, for  $w \in I$ ,

$$\begin{aligned} \frac{M_\xi(w)^2}{|\widehat{\xi}(w)|^2} &= \sum_{j \in 2\pi\mathbb{Z} \setminus 0} \frac{|\widehat{\xi}(w + j)|^2}{|\widehat{\xi}(w)|^2} = \sum_{j \in 2\pi\mathbb{Z} \setminus 0} |w/(w + j)|^{2k} |\widehat{\sigma}(w + j)/\widehat{\sigma}(w)|^2 \leq \\ &\frac{|w|^{2k}}{|\widehat{\sigma}(w)|^2} \sup_{j \in 2\pi\mathbb{Z} \setminus 0} \|(\cdot + j)^{-2k}\|_{L_\infty(I)} \sum_{j \in 2\pi\mathbb{Z} \setminus 0} |\widehat{\sigma}(w + j)|^2. \end{aligned}$$

Since  $\sigma \in C^\infty(\mathbb{R})$ ,  $\widehat{\sigma}$  is rapidly decaying, and therefore

$$\sum_{j \in 2\pi\mathbb{Z} \setminus 0} |\widehat{\sigma}(w + j)|^2 =: q(w)^2$$

converges to a smooth bounded function (the boundedness can be proved as follows: if we add to the sum the summand for  $j = 0$ , we get a periodic function which is bounded due to its continuity. Since each summand, including the  $j = 0$  one, is clearly bounded, it follows that the above sum is, too).

Since  $w \in I = [-t, t]$ ,  $\inf_{j \in 2\pi\mathbb{Z} \setminus 0} |w + j| = 2\pi - t$ , and we obtain the following estimate

$$(3.10) \quad \Lambda_\xi(w) \leq (2\pi - t)^{-k} |w|^k \|q/\widehat{\sigma}\|_{L_\infty(I)} =: \text{const} (2\pi - t)^{-k} |w|^k.$$

Consequently, since the Fourier transforms of  $f(2^{-k}\cdot)$  and  $g_k(2^{-k}\cdot)$  coincide on  $I$ , we have for  $k \geq r$

$$\begin{aligned}(2\pi)^{1/2} E_k(g_k) &= 2^{-k/2} \|f(\widehat{2^{-k}\cdot}) \Lambda_\xi\|_{L_2(I)} \\ &\leq \| |\cdot|^{-r} \Lambda_\xi \|_{L_\infty(I)} 2^{k/2} \| |\cdot|^r \widehat{f}(2^k\cdot) \|_{L_2(I)} \\ &\leq \text{const} (2\pi - t)^{-k} t^{k-r} \| |2^{-k}\cdot|^r \widehat{f}(\cdot) \|_{L_2(\mathbb{R})} \\ &\leq \text{const} (2\pi - t)^{-k} t^{k-r} 2^{-kr} \|f\|_{W_2^r}.\end{aligned}$$

Theorem 3.5 now follows when combining the last estimate (say, with  $t := 1$ ) for the projection error, with the estimate for the truncation error from Lemma 3.8.

Finally, for very smooth functions, better rates can be derived. First, upon substituting  $r = k$ ,  $t = \pi$  in the above estimate and in Lemma 3.8, we obtain:

**Corollary 3.11.** *If  $f \in W_2^k$  for some  $k$  then*

$$E_k(f) \leq \text{const} \pi^{-k} \|f\|_{W_2^k} 2^{-k^2}.$$

Concrete improvements of the rates of Theorem 3.5 require knowledge on the rate of growth of  $(\|f\|_{W_2^k})_{k \in \mathbb{Z}_+}$ . A typical example follows.

**Example 3.12.** Assume that  $\widehat{f}$  is so smooth so that its Fourier transform decays exponentially at  $\infty$ . Precisely, assume that  $|\widehat{f}(w)| \leq \text{const} e^{-\alpha|w|}$ , for some positive  $\alpha$ . Then  $\|f\|_{W_2^k} = O((k+1)!\alpha^{-k})$ , and therefore we conclude from the last theorem that

$$E_k(f) \leq \text{const} (k+1)! (\pi\alpha)^{-k} 2^{-k^2},$$

and therefore the error decays in this case exponentially in  $2^k$ .

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